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Multi-valued solutions of steady-state supersonic flow. Part 1. Linear analysis

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The shock wave equations for a perfect gas often provide more than one solution to a problem. In an attempt to find out which solution appears in a given physical situation, we present a linearized analysis of the equations of motion of a flow field with a shock boundary. It is found that a solution will be stable when there is supersonic flow downstream of the shock, and asymptotically unstable when there is subsonic flow downstream of it. It is interesting that both flows are found to be stable against disturbances of the d'Alembert type which grow from point sources; it is only when larger-scale line sources are considered that one can discriminate between the stabilities of the two types of flow. The results are applicable to supersonic flow over flat plates at incidence, to wedges, and to some cases of regular reflexion, diffraction and refraction of shocks.

1. Introduction

With given initial and boundary conditions, theory often results in more than one solution to a steady-state supersonic flow problem. For example, there are two solutions for a shock attached to a sharp wedge or cone and also for regular reflexion of a shock at a rigid wall, and sometimes there are three solutions for the wave confluence of a Mach reflexion, and four for regular refraction of a plane shock. Natural phenomena are of course single-valued, so the question to decide is the circumstances in which each member of a given set of solutions will appear in preference to the others in the real world.

One method of making the decision amounts to ordering the set on the basis of the rate of production of entropy $\dot{S}$ associated with each element of the set; that is, the first element has the smallest $\dot{S}$, the second the next smallest, and so on. When such sets are compared with experimental data it is very often found that it is the flow corresponding to the first element of the set which has appeared. It is tempting therefore to invoke *the principle of minimum entropy production*, which has had some success with non-convective systems (Donnelly, Hermann & Prigogine 1965; Meijer & Edwards 1970). In the present context many instances can be given where this principle works, namely wedge flow (Bleakney & Taub 1949), cone flow (Maccoll 1937), regular shock refraction in gases (Jahn 1956) and in metals (Laharrague, Morvan & Thouvenin 1968), some cases of shock reflexion and diffraction (Bleakney & Taub 1949; Mair 1952; Kawamura & Saito

1956), shock intersection (Smith 1959, 1962), and detonation (Zeldovich & Kompaneets 1960; Shchelkin & Troshin 1965). In the last instance the Chapman–Jouguet condition is one of minimum $\dot{S}$ and this is the flow which appears if only the stronger type of detonation is possible (Courant & Friedrichs 1948). Unfortunately numerous examples counter to the principle can also be given. One occurs at a plane rigid wall when there are simultaneous solutions† for regular and Mach reflexions. A regular reflexion then appears even if it has an $\dot{S}$ which is greater than those for any of the Mach-reflexion flows. Another counter example is provided by the phenomenon of weak detonation, for Zeldovich & Kompaneets (1960) have shown that this solution can appear in preference to the strong solution when there is a suitable distribution of sparks in the unburnt gas; it may also appear when there is efficient heat transfer – by radiation say – from the reaction zone; or it may appear for reactions which are initially exothermic but finally endothermic (Landau & Lifshitz 1959), and also for condensation discontinuities (Oswatitsch 1956; Landau & Lifshitz 1959). For all those phenomena which are of the weak detonation type, the Chapman–Jouguet condition is associated with $\dot{S}$ being a *maximum*. Hence the principle of minimum entropy production is an unsatisfactory criterion when convection is present.

Another method of making the decision is to argue that for an incident shock i of vanishingly small intensity the solution must be a continuation of one obtained from acoustic theory. More precisely, suppose that the amplitude of i declines continuously until i becomes a plane acoustic wave, then the element of the ordered set whose reflected wave has the same amplitude as the acoustic solution is the one chosen to be physically relevant. This approach has been used by Bleakney & Taub (1949), Polachek & Seeger (1951) and others. While this is successful in some cases, it fails in others. For example it fails to yield any result for Mach reflexion, while for regular refraction of shocks it is occasionally confounded by there being two solutions in the acoustic limit (Henderson & Macpherson 1968).

In the present paper (part 1) an attempt is made to formulate more satisfactory procedures for selecting the correct solution from the ordered set. To this end we first exploit an idea due to Landau & Lifshitz (1959), in which a direction is assigned to a wave by means of the vector which represents the component of the velocity parallel to it. This will often enable us to decide whether the given boundary conditions are sufficiently complete to support the flow; if they are not then the solution can be discarded from the set. Second, we study the stability of the remaining elements of the set with the help of the linearized equations of motion of the system. In order to keep the problem manageable attention is restricted to two-dimensional, adiabatic, steady-state flow of an inviscid perfect gas. It will be shown that a typical system of this type will be stable when there is supersonic flow ($V > a$) downstream of its leading shock, and asymptotically unstable when there is subsonic flow ($V < a$) downstream of it.

In a later paper (part 2) the flow will be treated as a purely mechanical system, and in a limited way this will allow some nonlinear effects to be taken into account.

† These conditions have been worked out by Bleakney & Taub for a diatomic gas with ratio of specific heats $\gamma = \frac{7}{5}$.

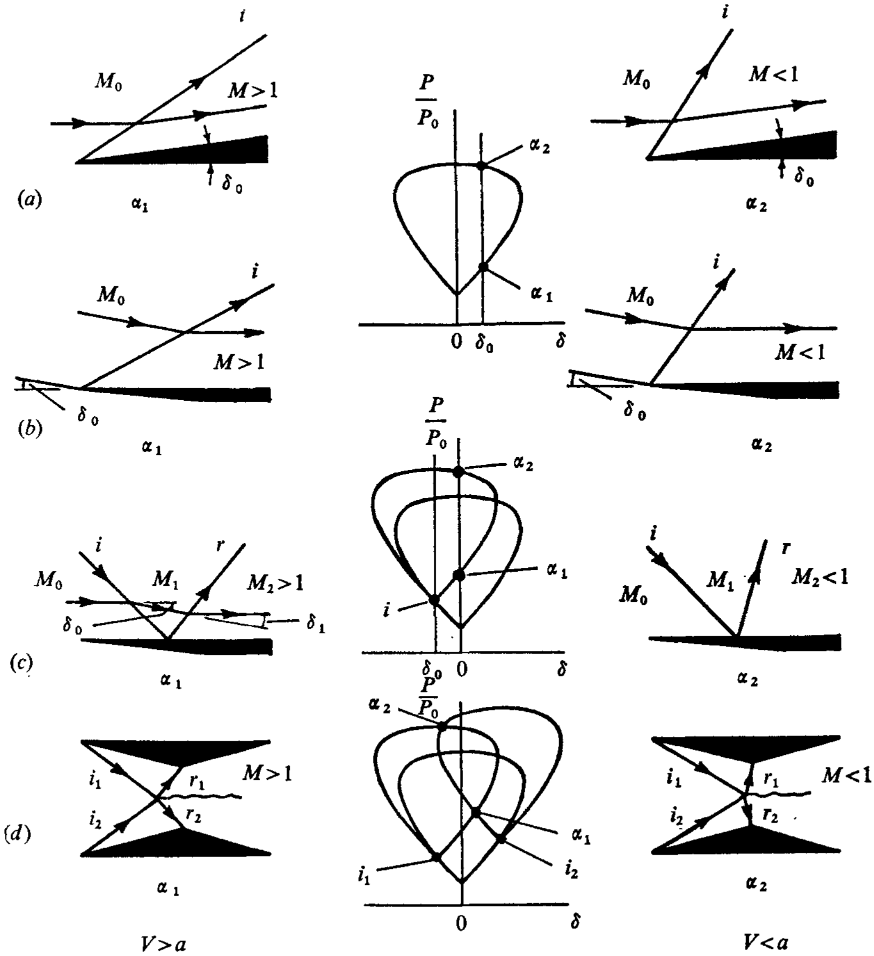

FIGURE 1. Flows with similar stability problems. (*a*) Wedge flow, $W \equiv \{\alpha_1, \alpha_2\}$. (*b*) Flat plate at incidence, $F \equiv \{\alpha_1, \alpha_2\}$. (*c*) Regular shock reflexion, $RF \equiv \{\alpha_1, \alpha_2\}$. (*d*) Regular shock intersection, $RI \equiv \{\alpha_1, \alpha_2\}$.

2. Flows similar to a flat plate at incidence

Supersonic wedge flow is illustrated in figure 1 (*a*) for both the physical and the shock-polar planes; the wedge angle is δ_0 and the free-stream Mach number is M_0. The ordered set of solutions will be written as $W \equiv \{\alpha_1, \alpha_2\}$, where α_1 is the weaker and smaller $\dot{S}$ solution and α_2 is the stronger and larger $\dot{S}$ one. The flow is supersonic downstream of the α_1 shock except for a small range of values of δ_0 near the maximum deflexion angle $\delta = \delta_{max}$. The α_2 shock always has subsonic flow downstream. Instead of treating the stability of wedge flow it is more convenient to treat the equivalent problem of a flat plate at an incidence angle δ_0 (figure 1 *b*), whose solution set is written as $F \equiv \{\alpha_1, \alpha_2\}$. Now if the stability of the flow can be found for each element of the ordered set, then not only will the result be known for the flat plate and the wedge, but it will be known for other flows as well.

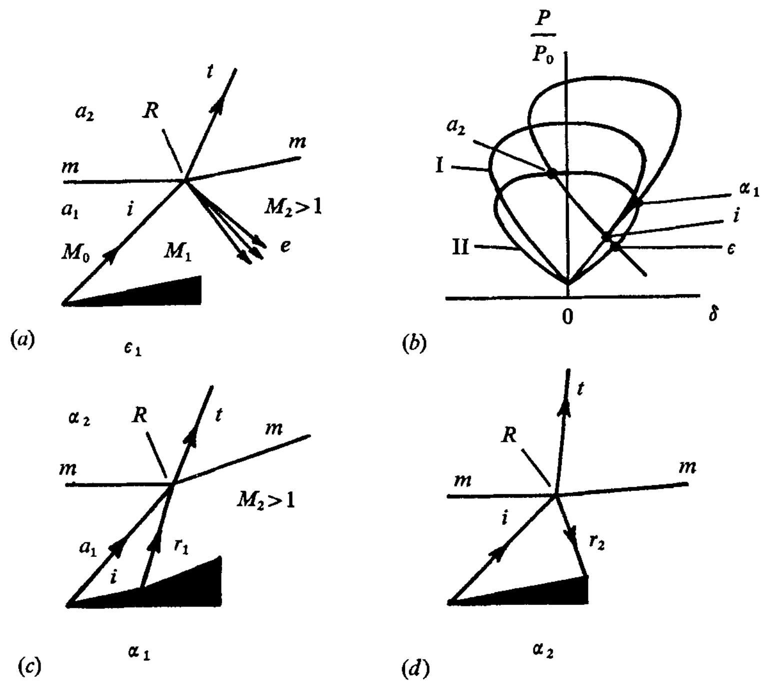

FIGURE 2. Regular refraction of a shock i at an interface mm, $RR \equiv \{\epsilon_1, \alpha_1, \alpha_2\}$.

For example, the ordered set for the regular reflexion of a plane shock at a rigid wall can be written as $RF \equiv \{\alpha_1, \alpha_2\}$ (figure 1 *c*), and it will be noticed from the diagrams that part of this flow can be regarded as that for a flat plate at an incidence angle $\delta_1 = -\delta_0$ and free-stream Mach number M_1. In this case the reflected shock r takes the place of the shock i in figure 1 (*b*). Similarly the regular intersection of two oblique shocks of either equal or unequal strengths can be treated in essentially the same way (figure 1 *d*), with $RI \equiv \{\alpha_1, \alpha_2\}$.

Wave direction

By making use of an idea due to Landau & Lifshitz (1959, p. 333), we can show that regular refraction of a plane shock can also be treated by the same methods. Now it is well known that when a gas crosses a shock the component of its velocity parallel to the wave is continuous, and the same applies to an expansion wave. We shall take the direction of this vector at any point to be the local direction of the wave. Furthermore we shall say that a wave *arrives* at any point which it intersects if the vector near the point and on the same side as the disturbance producing the wave is directed towards the point. We shall say that the wave *leaves* the point if the vector is directed away from it. A simple example is shown for regular reflexion in figure 1 (*c*), where the incident shock i arrives at the reflexion point and the reflected shock r leaves the same point.

As an example of the usefulness of this idea, consider regular refraction of a plane shock i at an interface between two different gases (figure 2). For slow–fast refraction† the solution set may be written as $RR \equiv \{\epsilon_1, \alpha_1, \alpha_2\}$ (Henderson 1966), where ϵ_1 represents the solution for a regular refraction with a reflected expansion wave e, and the $\alpha_{1,2}$ solutions have reflected shocks. For these three solutions the diagrams show that i arrives at the refraction point R and t leaves it. However, the reflected wave *leaves* R for the ϵ_1 and α_2 solutions but *arrives* at R for the α_1 solution. In the last case this means that t must be generated by i and r_1 intersecting at R on the interface between the two gases. Clearly, if this flow is to be physically possible an extra boundary in the form say of a wedge must be present in order to generate r_1, and the apex angle of this new wedge is predetermined by the polar diagram, and furthermore its position must be such as to cause r_1 to be focused exactly on the point R where i intersects the interface. This is a familiar situation in mechanics where the boundary conditions are often called 'too strict' for the system to exist in any but the most exceptional circumstances. Such systems appear as transitions between systems of different types. In any event, without the very special boundary conditions just mentioned the α_1 solution cannot exist physically and it may be discarded from the set, so that $RR_2 \equiv \{\epsilon_1, \alpha_2\}$. The polar diagram shows that for the ϵ_1 solution the flow is supersonic downstream of both t and e, so that its stability problem should be the same as that for the α_1 elements of the sets W, F, RF and RI. By contrast, for the α_2 element of RR_2, the flow is always subsonic downstream of t and it may be either subsonic or supersonic downstream of r_2. However, if t can be destabilized by some disturbance, then r_2 will also be destabilized, so the stability of this solution should be similar to that of the α_2 elements of W, F, RF and RI.

3. The equations of motion

To avoid unbounded velocities far downstream, the wall boundary is assumed to be of large but finite length. The flow field to be considered for the α_1 case is a supersonic triangular region bounded by the wall, the shock and the first Mach line which reaches the shock from the downstream edge of the wall. The flow field is subsonic for the α_2 case and is again bounded by the wall and the shock, but now has a sonic surface joining the downstream edge to the shock. This is followed immediately by a Prandtl–Meyer expansion, and the last of its waves to meet the sonic line terminates the flow field to be considered, as discussed by Guderley (1947, 1962).

Referring to figure 3, the equations of motion for unsteady, two-dimensional, adiabatic flow of an inviscid fluid are as follows (Landau & Lifshitz 1959, pp. 2–4), with velocity components (u, v), density ρ, pressure P, speed of sound a, resultant velocity V and entropy S.

Continuity

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x}+\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{\rho}\frac{\partial \rho}{\partial y} = 0. \tag{1}$$

† A slow–fast refraction is one for which the speed of sound a_{I} in the incident-shock medium is less than that (a_{II}) in the transmitted-shock medium, i.e. $a_{\text{I}} < a_{\text{II}}$. Conversely a fast–slow refraction is defined by $a_{\text{I}} > a_{\text{II}}$.

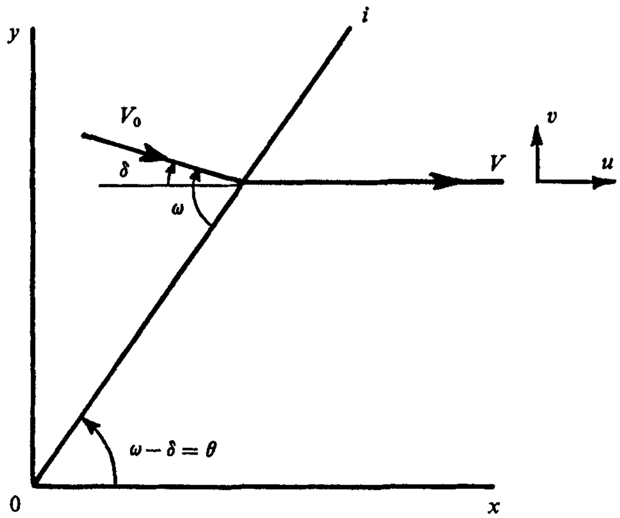

FIGURE 3. Nomenclature.

Euler

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}=-\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}-\frac{1}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho\frac{\partial S}{\partial x}, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}=-\frac{a^2}{\rho}\frac{\partial \rho}{\partial y}-\frac{1}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho\frac{\partial S}{\partial y}. \tag{3}$$

Energy

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}=0. \tag{4}$$

Potential

$$u=\partial\Phi/\partial x,\quad v=\partial\Phi/\partial y. \tag{5}$$

Then (2) and (3) become

$$\Phi_{xt}+\Phi_x\Phi_{xx}+\Phi_y\Phi_{yx}=-\frac{a^2}{\rho}\rho_x-\frac{1}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho S_x, \tag{6}$$

$$\Phi_{yt}+\Phi_x\Phi_{xy}+\Phi_y\Phi_{yy}=-\frac{a^2}{\rho}\rho_y-\frac{1}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho S_y. \tag{7}$$

Integrating (7) with respect to y gives

$$\Phi_t+\tfrac{1}{2}\Phi_x^2+\tfrac{1}{2}\Phi_y^2+\int_{\rho_0}^{\rho}\frac{a^2}{\rho}d\rho+\int_{S_0}^{S}\left(\frac{\partial P}{\partial S}\right)_\rho\frac{ds}{\rho}=f(x,t), \tag{8}$$

where $f(x,t)$ is an arbitrary function of (x,t). Now (8) can also be obtained by integrating (6) with respect to x, but this leads to the arbitrary function $f(y,t)$, and therefore $f(x,t)=f(y,t)=f(t)$. Hence $f(t)$ can be removed from the equations by the substitution $\hat{\Phi}=\Phi-\int f(t)\,dt$. We do this and then drop the circumflex on Φ, with the net result that $f(x,t)$ can be set equal to zero. Now differentiating (8) with respect to t we get

$$\Phi_{tt}+\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt}+\frac{a^2}{\rho}\rho_t+\frac{1}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho S_t=0. \tag{9}$$

Next we substitute (5)–(7) and (9) into (1), to obtain behind the shock

$$(a^2-\Phi_x^2)\,\Phi_{xx}+(a^2-\Phi_y^2)\,\Phi_{yy}-2\Phi_x\Phi_y\Phi_{xy}-\Phi_{tt}-2\Phi_x\Phi_{xt}-2\Phi_y\Phi_{yt}=0. \quad (10)$$

The entropy terms for (10) are $-\rho^{-1}(\partial P/\partial S)_\rho\,[S_t+\Phi_x S_x+\Phi_y S_y]$ but these vanish by virtue of (4). The boundary conditions for this equation are

$$\Phi_y = 0 \quad \text{on} \quad y=0, \quad (11)$$

that is, the velocity is parallel to the surface, and

$$\Phi_x\cos\theta+\Phi_y\sin\theta = V_0\cos\omega \quad \text{on the shock}, \quad (12)$$

which is the condition for continuity across the shock of the component of the velocity parallel to the wave. Now an oblique shock is completely determined by two independent parameters and it will be convenient to take these to be M_0 and δ. Then

$$\omega = \Omega(M_0,\delta), \quad \theta = \Omega(M_0,\delta)-\delta. \quad (13)$$

In principle, the form of the function Ω may be obtained from equation (A 2) of the appendix. It is a smooth bounded function of (M_0,δ) outside a small neighbourhood of $\delta=\delta_{0\,\mathrm{max}}$.

The next step is to linearize these equations by substituting $\langle\Phi\rangle+\epsilon\phi$, $\langle a\rangle+\epsilon a'$, $\langle V_0\rangle+\epsilon V_0'$, $\langle\omega\rangle+\epsilon\omega'$, $\langle\theta\rangle+\epsilon\theta'$ and $\langle M_0\rangle+\epsilon M_0'$ for Φ, a, V_0, ω, θ and M_0 respectively, and neglecting higher powers of ϵ. Here angular brackets indicate mean or zeroth-order components, and ϵ is a small parameter. Taking $\epsilon=0$, the mean equations have solutions of the form $\langle\phi\rangle = Vx$, where V is constant and $\langle a\rangle$ and $\langle\theta_0\rangle$ are also constant, i.e. invariant in space and time. There are two such solutions, namely α_1, with $M = V/\langle a\rangle > 1$, and α_2, with $M<1$. Substituting these solutions into (10)–(13) and retaining only terms of first order in ϵ, the linearized perturbation equations are

$$(1-M^2)\,\phi_{xx}+\phi_{yy}-\phi_{\tau\tau}-2M\phi_{x\tau}=0, \quad (14)$$

$$\phi_y = 0 \quad \text{on} \quad y=0, \quad (15)$$

$$\phi_x\cos\langle\theta\rangle+\phi_y\sin\langle\theta\rangle = V_0'\cos\langle\omega\rangle-\omega'\langle V_0\rangle\sin\langle\omega\rangle + \theta' V\sin\langle\theta\rangle \quad \text{on} \quad y = x\tan\langle\theta\rangle, \quad (16)$$

where

$$\omega' = \frac{\partial\Omega}{\partial M_0}M_0'+\frac{\partial\Omega}{\partial\delta}\delta', \quad \theta' = \frac{\partial\Omega}{\partial M_0}M_0'+\left(\frac{\partial\Omega}{\partial\delta}-1\right)\delta' \quad (17)$$

and $\tau\equiv\langle a\rangle t$ is a normalized time variable.

4. Solutions of the linearized equations of motion and their stability

D'Alembert-type solutions ϕ_A

Such solutions are well known for (14); see for example Sears (1954, p. 116). They are

$$\phi_A = \sigma^{-1}\,[f(\sigma+\beta^2\tau-Mx)+g(\sigma-\beta^2\tau+Mx)], \quad (18)$$

where

$$\sigma\equiv(x^2-\beta^2y^2)^{\frac{1}{2}}, \quad \beta^2\equiv M^2-1. \quad (19), (20)$$

Making use of these results it is straightforward to construct solutions ϕ_s for sinusoidal oscillations in the flow, namely

$$\phi_s = \frac{k}{\sigma} \exp i\nu \left[\tau - \frac{M}{\beta^2} x \pm \frac{\sigma}{\beta^2}\right], \tag{21}$$

where k is a real constant and ν is a real positive constant. Now if (14) is to describe real disturbances which propagate from a point into the flow it must have real characteristics, and by inspection of (18) we see that this requires that σ should be real for both the α_1 ($V > a$) and the a_2 ($V < a$) flow. It follows from (21) that ϕ_s will be bounded for all (x, y, t) except $(0, 0, t)$. Hence both the α_1 and the α_2 flow will experience conservative oscillations when these disturbances emanating from a point in the x, y plane are present, and we conclude that both flows are stable against them. This approach therefore fails to discriminate between the stabilities of the α_1 and α_2 flows, so we shall now investigate the effect that larger-scale line disturbances have on the two flows.

Line-type solutions

If a solution of the form

$$\phi(x, y, \tau) = Y(y)\, E(x, \tau) \tag{22}$$

is assumed, then (14) separates, giving

$$Y_{yy} + \lambda^2 Y = 0 \tag{23}$$

and

$$\beta^2 E_{xx} + 2ME_{x\tau} + E_{\tau\tau} + \lambda^2 E = 0, \tag{24}$$

where λ is a constant. Boundary condition (15) becomes

$$Y_y = 0 \quad \text{at} \quad y = 0, \tag{25}$$

so that the solution to (23) is

$$Y(y) = \cos \lambda y. \tag{26}$$

Thus boundary condition (16) becomes

$$E_x \cos \lambda y \cos \langle\theta\rangle - \lambda E \sin \lambda y \sin \langle\theta\rangle = V_0' \cos \langle\omega\rangle - \omega' \langle V_0\rangle \sin \langle\omega\rangle + \theta' V \sin \langle\theta\rangle \tag{27}$$

on $y = x \tan \langle\theta\rangle$. This is of the form

$$E_x \cos \langle\theta\rangle \cos (\lambda x \tan \langle\theta\rangle) - E \sin \langle\theta\rangle \sin (\lambda x \tan \langle\theta\rangle) = h(x, \tau), \tag{28}$$

where $h(x, \tau)$ is an arbitrary function of x and τ bounded if V_0', ω' and θ' are bounded.

Equation (24) has solutions of the form

$$E(x, \tau) = A \exp [i\lambda\beta^{-1}(x - 2M\tau)], \tag{29}$$

where the physical solution is the real part of this quantity. This particular solution, or linear combinations of it with different values of λ, will not in general satisfy (28). If it is assumed that the perturbing function $h(x, \tau)$ is such that it does, then the left side of (28) becomes

$$\lambda A \exp [i\lambda\beta^{-1}(x - 2M\tau)] \{i\beta^{-1} \cos \langle\theta\rangle \cos (\lambda x \tan \langle\theta\rangle) + \sin \langle\theta\rangle \sin (\lambda x \tan \langle\theta\rangle)\}. \tag{30}$$

Now on account of (26), λ must be real, for otherwise ϕ (and therefore u and v) will increase indefinitely as y increases. Thus for given non-zero β and $\langle\theta\rangle$, the quantity in the braces in (30) satisfies

$$K_1 \leqslant |\{\,\}| \leqslant K_2,$$

where $K_{1,2}$ are positive constants. Thus the perturbing function $h(x, \tau)$ remains bounded if and only if the same is true of $\exp[i\lambda\beta^{-1}(x - 2M\tau)]$. This will be true both as $x \to \infty$ and as $\tau \to \infty$ only if λ/β and therefore β is real. But by (20) this means that the Mach number downstream of the shock must be greater than one.

Thus for the α_1 shock, which has supersonic flow downstream, the flow experiences conservative oscillations when the shock does so, and we conclude that the flow is *stable*. On the other hand in the α_2 case M is always less than one, and conservative oscillations of the shock lead to oscillations in the flow downstream of it which increase exponentially with time or with distance. We conclude that the α_2 flow is *asymptotically unstable*.

Remark on the stability of a normal shock

In the above analysis, we have assumed that the downstream edge of the plate is so remote that any disturbances arising from it do not affect the stability of the shock at the upstream (leading) edge. For small disturbances, this assumption is necessary only for the α_2 flow, where $M < 1$. However, we have also assumed that the plate is of finite length; therefore the α_2 shock intersects the sonic surface which propagates from the downstream edge at some finite value of y. In the limit as $\delta \to 0$, the α_2 shock approaches the normal shock condition and the intersection moves towards infinity in the y direction. Thus both the shock and the sonic line become normal in the semi-infinite region above the plate. If it could be assumed that the stability of the shock near the wall still depends only on perturbations in the same region (i.e. that the boundary conditions at infinity are not critical), then our analysis (with minor changes) would still apply in the limit, so that we could conclude that a normal shock is asymptotically unstable in a semi-infinite region. Of course, there is no way of testing this result by experiment.

The evidence gained from experiment indicates that a shock which is *everywhere* normal to the flow cannot be set up when there is a downstream sonic surface, although it is possible for it to be locally normal at particular points. The only way which seems practicable to attain a stable shock which is closely normal everywhere is to replace the sonic surface by a piston, or its equivalent, acting between parallel walls. To apply this to our problem we introduce a second plate into the flow, which is placed above the first one and parallel to it, and a piston moves between them. Then there will exist a normal stationary shock between the plates if the piston withdraws downstream at the correct velocity. This velocity is equal to the difference in the gas velocities on either side of the shock. There is no *a priori* reason why the stability of a shock between two walls should be the same as that of a similar shock in a semi-infinite region.

Form of the perturbation

To show the form of perturbation required in order that the solution should have the form

$$\phi(x, y, \tau) = A \exp[i\lambda\beta^{-1}(x - 2M\tau)] \cos \lambda y \tag{31}$$

given by (22), (26) and (29), consider the case where the perturbations $\epsilon V_0'$ and $\epsilon M_0'$ are zero but the perturbation $\epsilon\delta'$ is non-zero, that is conditions upstream of the shock remain constant but the wedge angle is varied. Then

$$h(x, \tau) = \langle V_0 \rangle \left[\sin\langle\omega\rangle \left\langle\frac{\partial\Omega}{\partial\delta}\right\rangle + \frac{V}{\langle V_0\rangle} \sin\langle\theta\rangle \left(\left\langle\frac{\partial\Omega}{\partial\delta}\right\rangle - 1\right)\right] \delta', \tag{32}$$

where according to equations (A 1) and (A 5) of the appendix

$$\frac{V_0}{V} = \left[1 - 4\frac{(M_0^2 \sin^2\omega - 1)(\gamma M_0^2 \sin^2\omega - 1)}{(\gamma+1)^2 M_0^4 \sin^4\omega}\right]^{\frac{1}{2}}, \tag{33}$$

$$\frac{\partial\Omega}{\partial\delta} = -\frac{\sin\omega\cos\omega}{\sin\delta\cos\delta}\left[1 - \frac{(\gamma+1) M_0^4 \sin^2\cos^2\omega}{(M_0^2\sin^2\omega - 1)[1 + \frac{1}{2}(\gamma+1)M_0^2 - M_0^2\sin^2\omega]}\right]^{-1}. \tag{34}$$

All of the quantities in the last two equations are mean components, the angular brackets being omitted for simplicity. Hence if δ' has the form

$$\delta'(x, \tau) = \exp[i\lambda\beta^{-1}(x - 2M\tau)] \times \{\sin\langle\theta\rangle \sin(\lambda x \tan\langle\theta\rangle) + i\beta\cos\langle\theta\rangle \cos(\lambda x \tan\langle\theta\rangle)\}, \tag{35}$$

that is a travelling wave of velocity $2V$, wavenumber λ/β and amplitude varying with x, then the potential behind the shock, once the transients have died out, will be given by (30) with

$$A = \frac{\langle V_0\rangle}{\lambda}\left[\sin\langle\omega\rangle\left\langle\frac{\partial\Omega}{\partial\delta}\right\rangle + \frac{V}{\langle V_0\rangle}\sin\langle\theta\rangle\left(\left\langle\frac{\partial\Omega}{\partial\delta}\right\rangle - 1\right)\right]. \tag{36}$$

5. Guderley boundary conditions

It is natural to inquire if the α_2 flow can be stabilized by placing an extra boundary in the flow in the manner suggested by Guderley (1947, 1962) for transonic flow. In this case it is more convenient to discuss a wedge flow rather than a plate at incidence. Then the Guderley boundary takes the form of a second wedge; see figure 4. The apex angle δ_1 of the second wedge must be such that it exceeds the shock attachment angle for M_0 ($\delta_1 > \delta_{max}$). Guderley considers continuous changes in a boundary parameter, namely the ratio $l_0/l_1 \equiv l_{01}$ of the lengths of the two wedges. Initially the shock will be detached if $l_{01} \to 0$, so that the apex of the second wedge is very close to that of the first. The parameter l_{01} is now continuously increased so that the second wedge moves downstream, and this will cause the shock to approach the apex of the first wedge. According to Guderley an α_2 shock will appear at this apex at the attachment condition $l_{01} = l_a$. Any further increase in l_{01} will cause the shock to pass this apex, resulting in the three-shock system shown in figure 4. There will now be an α_1 shock i on the apex

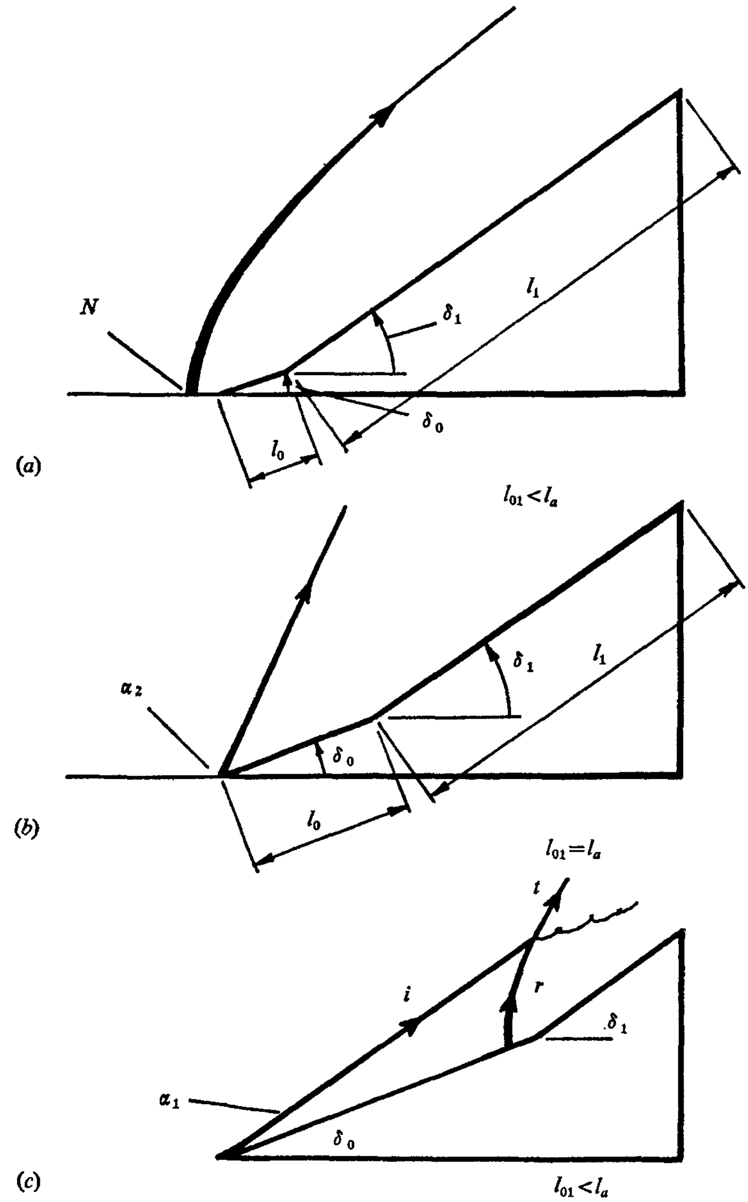

FIGURE 4. The Guderley boundary.

of the first wedge, a detached shock r off the apex of the second wedge and a third shock t which arises through the interaction of the other two. This wave system persists as l_{01} becomes indefinitely large. We therefore conclude that on the continuum $0 < l_{01} < \infty$ there is at most only one value of l_{01} ($= l_a$) for which the α_2 shock may appear on the apex of the first wedge, whereas there is a detached shock off the wedge for the continuum $0 < l_{01} < l_h$ and a three-shock system for the continuum $l_a < l_{01} < \infty$. Hence the boundary conditions for the α_2 shock exist only as a transition between two other systems, and as in the case of the

α_1 refraction in RR_1, the boundary condition is 'too strict', and the α_2 shock can appear only during the transition.

Even this last conclusion is at most true only for the transonic flows considered by Guderley. At larger Mach numbers it is easy to show that there are discontinuous pressure changes during the transition. Thus the bow shock is normal to the boundary streamline at N (figure 4), so the local pressure ratio across the shock at N is

$$\left.\frac{P}{P_0}\right|_N = \frac{2\gamma}{\gamma+1}M_0^2 - \frac{\gamma-1}{\gamma+1}. \tag{37}$$

When there is an α_2 shock at the first wedge's apex, the pressure ratio is

$$\left.\frac{P}{P_0}\right|_{\alpha_2} = \frac{2\gamma}{\gamma+1}M_0^2\sin^2\omega_{\alpha_2} - \frac{\gamma-1}{\gamma+1}, \tag{38}$$

while the pressure ratio across the combined α_1 and bow shock for the three-shock system is, at the bounding streamline along the first wedge,

$$\left.\frac{P}{P_0}\right|_{\alpha_1 Xr} = \left(\frac{2\gamma}{\gamma+1}M_0^2\sin^2\alpha_1 - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{2\gamma}{\gamma+1}M^2 - \frac{\gamma-1}{\gamma+1}\right). \tag{39}$$

It is clear that in general we have along the bounding streamline

$$\left.\frac{P}{P_0}\right|_N \neq \left.\frac{P}{P_0}\right|_{\alpha_2} \neq \left.\frac{P}{P_0}\right|_{\alpha_1 \times r}, \tag{40}$$

so that there are discontinuities in the pressure forces during transition. For an ideal inviscid gas this means discontinuous accelerations of the gas, so even with the Guderley boundary the α_2 flow is unstable, and stability is only approached in the limit $M_0 \to 1$, in the Guderley sense.

Appendix. The sign of $[\partial(V/V_0)/\partial\delta]_{M_0}$ for α_1 and α_2 flow

From Ames (1953),

$$\left(\frac{V}{V_0}\right)^2 = 1 - 4\frac{(M_0^2\sin^2\omega - 1)(\gamma M_0^2\sin^2\omega - 1)}{(\gamma+1)^2 M_0^4\sin^4\omega} \tag{A 1}$$

and

$$\cot\delta\cot\omega = \frac{\frac{1}{2}(\gamma+1)M_0^2}{M_0^2\sin^2\omega - 1} - 1. \tag{A 2}$$

Eliminating $M_0\sin\omega$ between (A 1) and (A 2) we get after some reduction

$$\frac{V}{V_0} = \frac{1+\frac{1}{2}(\gamma-1)M_0^2 + \cot\delta\cot\omega}{1+\frac{1}{2}(\gamma+1)M_0^2 + \cot\delta\cot\omega}. \tag{A 3}$$

Now obtain $[\partial(V/V_0)/\partial\delta]_{M_0}$ from (A 3). The result is

$$\frac{V_0}{V}\left(\frac{\partial V/V_0}{\partial\delta}\right)_{M_0} = -\frac{\operatorname{cosec}^2\delta\cot\omega + \operatorname{cosec}^2\omega\cot\delta\,\partial\omega/\partial\delta}{[1+\frac{1}{2}(\gamma-1)M_0^2+\cot\delta\cot\omega][1+\frac{1}{2}(\gamma+1)M_0^2+\cot\delta\cot\omega]}. \tag{A 4}$$

The denominator of (A 4) is always positive; only the numerator N(A 4) can change sign. In order to develop an expression for it, obtain $(\partial\omega/\partial\delta)_{M_0}$ from (A 2), that is

$$\left(\frac{\partial\omega}{\partial\delta}\right)_{M_0} = -\frac{\sin\omega\cos\omega}{\sin\delta\cos\delta}\left[1-\frac{(\gamma+1)\,M_0^4\sin^2\omega\cos^2\omega}{(M_0^2\sin^2\omega-1)\,[1+\frac{1}{2}(\gamma+1)\,M_0^2-M_0^2\sin^2\omega]}\right]^{-1}. \tag{A 5}$$

Substituting (A 5) into the numerator of (A 4) we get

$$\mathrm{N}(\mathrm{A}\,4) = -\operatorname{cosec}^2\delta\cot\omega \times\left[\frac{(\gamma+1)\,M_0^4\sin^2\omega\cos^2\omega}{\gamma M_0^4\sin^4\omega+2[1-\frac{1}{4}(\gamma+1)\,M_0^2]\,M_0^2\sin^2\omega-[1+\frac{1}{2}(\gamma+1)\,M_0^2]}\right]. \tag{A 6}$$

Only the denominator of (A 6) may change sign, and this happens when it is zero, that is when

$$\gamma M_0^2\sin^2\omega = \tfrac{1}{4}(\gamma+1)\,M_0^2-1+\{(\gamma+1)\,[1+\tfrac{1}{2}(\gamma-1)\,M_0^2+\tfrac{1}{16}(\gamma+1)\,M_0^4]\}^{\frac{1}{2}}. \tag{A 7}$$

But this is just the condition $\delta = \delta_{\max}$ for there to be maximum streamline deflexion across an oblique shock wave (Ferri 1949, p. 47), that is, where the α_1 and α_2 branches join on the shock polar. We can now determine the sign of $[\partial(V/V_0)/\partial\delta]_{M_0}$ for α_1, by allowing $M_0\sin\omega\to 1$ as for a weak shock. It follows that N(A 4) > 0, and therefore from (A 4)

$$\left(\frac{\partial V/V_0}{\partial\delta}\right)_{M_0} < 0 \quad \text{for the } \alpha_1\ (V > a) \text{ flow.} \tag{A 8}$$

Conversely, for the α_2 flow we allow $M_0\sin\omega\to M_0$ (normal shock), and this shows that N(A 4) < 0, and hence

$$\left(\frac{\partial V/V_0}{\partial\delta}\right)_{M_0} > 0 \quad \text{for the } \alpha_2\ (V < a) \text{ flow.} \tag{A 9}$$

Hence a small quasi-static increase in δ decreases the velocity behind the shock for the α_1 flow. It simultaneously increases the pressure, as can be seen from the polar. This will provide a restoring force which will be opposed to the change in δ. This supports the idea that the α_1 flow is stable. The converse is true for the α_2 flow. We take these considerations up in detail in part 2.

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